Problem sheet 1

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1 Early warfare

Originally[1], warfare was conducted through hand to hand combat. Thus, a single person had the ability to despatch only one other person at a time. Namely, the combat was *one on one* until one person was unable to fight any longer. Consider two opposing armies of size $A(t)$ and $B(t)$ and suppose $A(0) = A_0$, $B(0) = B_0$, where $A_0 > B_0$.

1. Suppose the armies are equally matched such that when members of the A and B armies meet they are both equally like to win the fight. Further, assume that the interaction rate constant between the two armies is r, write the above combat interaction down as two interaction equations.

2. Show that the Law of Mass Action can be used to convert the early warfare interactions into the following ODEs,

$$\dot{A} = -rAB, \quad A(0) = A_0, \tag{1}$$
$$\dot{B} = -rAB, \quad B(0) = B_0. \tag{2}$$

3. Derive the steady states of equations (1) and (2). What happens when you try to characterise their stability?

4. Show that

$$A(t) = B(t) + \text{Constant}, \tag{3}$$

   where you should define the constant explicitly.

5. In question 3 you should have found that there was an infinite number of steady states. Show how equation (3) collapses all possibilities to just one.

6. Construct the (A, B) phase plane and use it to characterise the steady states stability.

7. Show that equations (1) and (2) can be rewritten as

$$\dot{A} = -rA(A - A_0 + B_0), \tag{4}$$
$$\dot{B} = -rB(B + A_0 - B_0). \tag{5}$$

8. Derive the steady states of equations (4) and (5).

9. Considering equations (4) and (5) as separate scalar ODEs characterise the stability of the previously derived steady states analytically and graphically through constructing the phase planes $(A, \dot{A})$ and $(B, \dot{B})$.

10. Consider equations (4) and (5) as a system and characterise the stability of the steady states algebraically and graphically by plotting the (A, B) phase plane.

11. Do the results from questions (9) and (10) correspond to the results from question 6?

12. Solve equations (4) and (5) explicitly and sketch the resulting $(t, A(t))$ and $(t, B(t))$ curves on the same axis.

13. You have now derived the same result in three different ways. Without referring to the mathematics, interpret your result.

[1] This question is an adapted version of Lanchester's Laws. Have a look into how modern warfare differs from early warfare.

1.1 Answers

1. Since combat is one on one then one person from the A side will interact with one person from the B side, thus, the interaction term is $A+B$. Since the fight happens until one of the combatants can no longer compete there are two possible outcomes, either A wins or B wins. Thus the two interaction equations are

$$A + B \xrightarrow{r} A, \tag{6}$$
$$A + B \xrightarrow{r} B. \tag{7}$$

2. Equations (1) and (2) follow directly from applying the Law of Mass Action to equations (6) and (7).

3. There are two infinite families of steady states, of the form $\left(\hat{A}, 0\right)$ and $\left(0, \hat{B}\right)$ for any constant values $\hat{A}$, or $\hat{B}$. The Jacobian of the system is

$$J = \begin{bmatrix} -rB & -rA \\ -rB & -rA \end{bmatrix}, \tag{8}$$

and

$$J(\hat{A}, 0) = \begin{bmatrix} 0 & -r\hat{A} \\ 0 & -r\hat{A} \end{bmatrix}, \quad J(0, \hat{B}) = \begin{bmatrix} -r\hat{B} & 0 \\ -r\hat{B} & 0 \end{bmatrix}. \tag{9}$$

In both cases one eigenvalue is negative and the other is zero. Thus, the linear analysis is inconclusive.

4. Subtracting equation (2) from equation (1) provides

$$\dot{A} - \dot{B} = 0. \tag{10}$$

Integrating both sides of equation (10) with respect to time provides

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t'}(A - B)dt' = 0, \tag{11}$$

which upon simplification and rearranging is

$$A(t) = B(t) - B_0 + A_0. \tag{12}$$

5. Question 3 led us to derive that $\left(\hat{A}, 0\right)$ and $\left(0, \hat{B}\right)$ are steady states for all constant values of $\hat{A}$ and $\hat{B}$. However, we also have to satisfy equation (12). Since $A_0 > B_0$ we must have that $\lim_{t\to\infty} B(t) = 0$ and $\lim_{t\to\infty} A(t) = \hat{A} = A_0 - B_0$.

6. The (A, B) phase plane is given in Figure 1. We observe that the horizontal and vertical axes are the nullclines and the steady states. Further, in the positive (A, B) quadrant both A and B are decreasing. However, the trajectories must travel along a line of $B = A-$Constant as derived earlier. The initial conditions pick exactly one of these lines and the trajectory heads to $\left(0, \hat{A}\right)$ as derived. Since we must remain on this line and the populations cannot go negative $\left(0, \hat{A}\right)$ is stable.

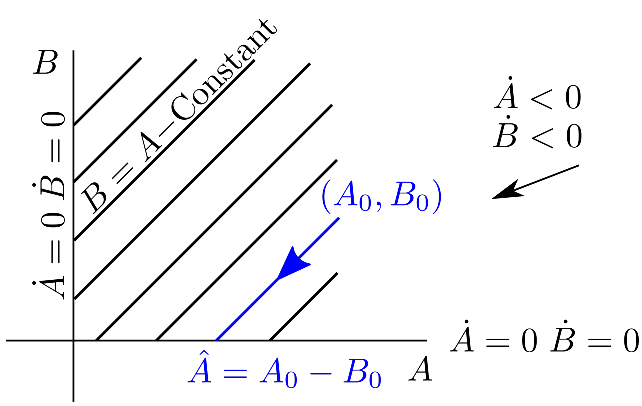

Figure 1: (A, B) phase plane of equations (1) and (2).

7. Substituting equation (12) into equations (1) and (2) gives equations (4) and (5).

8. The steady states are $(0,0)$, $(0, B_0 - A_0)$, $(A_0 - B_0, 0)$ and $(A_0 - B_0, B_0 - A_0)$.

9. We first note that $A_0 - B_0 > 0$. From the last question the steady states of

$$\dot{A} = -rA(A - A_0 + B_0) = f(A) \tag{13}$$

are $A_{st} = 0, A_0 - B_0$. Deriving

$$f'(A) = -2rA - r(B_0 - A_0) \tag{14}$$

and substituting in the steady states give

$$f'(0) = -r(B_0 - A_0) > 0, \tag{15}$$
$$f'(A_0 - B_0) = -r(A_0 - B_0) < 0. \tag{16}$$

Thus, $A_{st} = 0$ is unstable and $A_{st} = A_0 - B_0$ is stable.

Similarly, if $g(B) = -rB(B + A_0 - B_0)$ and the steady states are $B_{st} = 0, B_0 - A_0$ then

$$g'(B) = -2rB - r(A_0 - B_0) \tag{17}$$

and

$$g'(0) = -r(A_0 - B_0) < 0 \tag{18}$$
$$g'(B_0 - A_0) = -r(B_0 - A_0) > 0. \tag{19}$$

Thus, $B_{st} = 0$ is stable and $B_{st} = B_0 - A_0$ is unstable. Also note $B_{st} = B_0 - A_0 < 0$, which is unphysical and so we do not really need to consider it.

The phase planes of $\left(A, \dot{A}\right)$ and $\left(B, \dot{B}\right)$ are shown in Figure 2

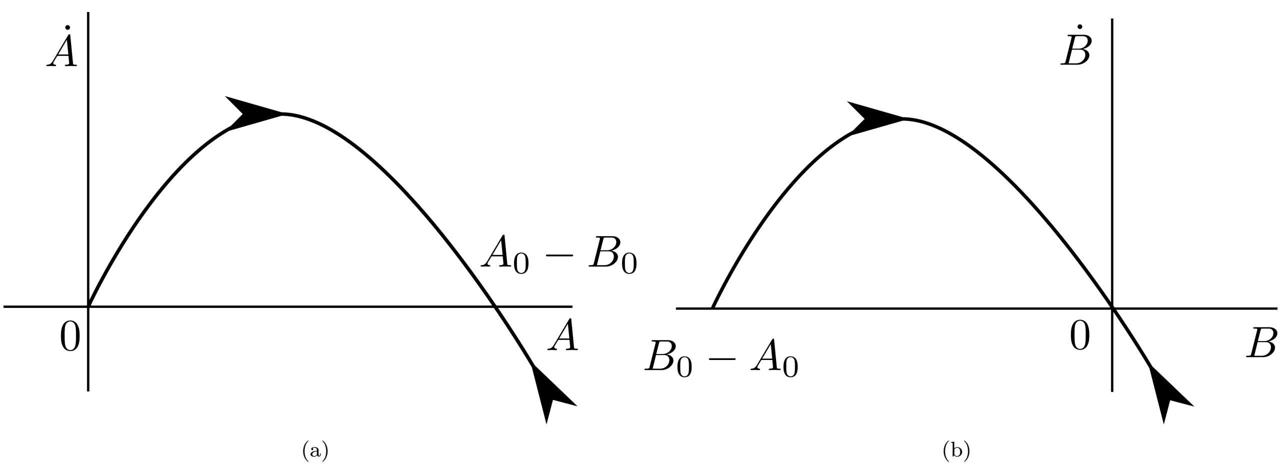

Figure 2: Phase planes of equations (4) and (5).

10. The Jacobian of equations (4) and (5) is

$$J_2 = \begin{bmatrix} -2rA - r(B_0 - A_0) & 0 \\ 0 & -2rB - r(A_0 - B_0) \end{bmatrix}, \tag{20}$$

and, so,

- the eigenvalues of $J_2(0,0)$ are $\lambda_{1,2} = \pm r(B_0 - A_0)$, thus, $(0,0)$ is a saddle point;
- the eigenvalues of $J_2(A_0 - B_0, 0)$ are $\lambda_{1,2} = -r(A_0 - B_0) < 0$, thus, $(A_0 - B_0, 0)$ is a stable node;
- the eigenvalues of $J_2(0, B_0 - A_0)$ are $\lambda_{1,2} = -r(B_0 - A_0) > 0$, thus, $(0, B_0 - A_0, 0)$ is an unstable node;
- the eigenvalues of $J_2(A_0 - B_0, B_0 - A_0)$ are $\lambda_{1,2} = \pm r(A_0 - B_0)$, thus, $(A_0 - B_0, B_0 - A_0)$ is a saddle point.

The accompanying phase plane is shown in Figure 3.

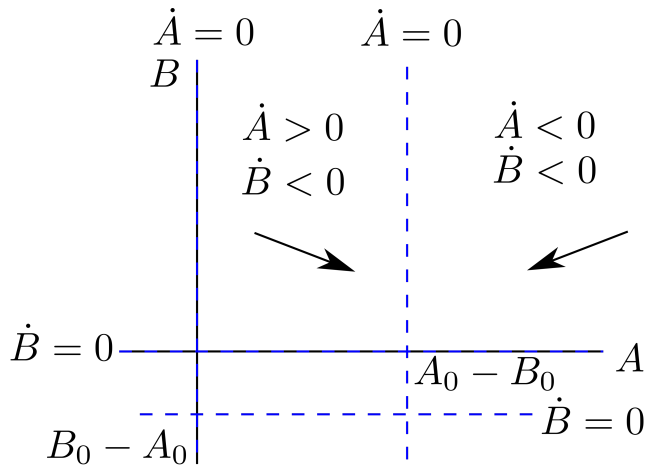

Figure 3: Phase plane of equations (4) and (5).

11. When we analysed system (1)-(2) initially we found that the linear analysis approach did not work. However, converting system (1)-(2) to system (4)-(5) does allow us to use the algebraic techniques. However, converting from (1)-(2) to system (4)-(5) does introduce a number of "artificial" steady states. However, these are all unstable and in all cases we reproduce that the system tends to $(0, A_0 - B_0)$.

12. Let us, first, solve equation (4). Using partial fractions we get that equation (4) is equivalent to

$$\int_{A_0}^{A(t)} -\frac{1}{(A_0 - B_0)\,A} + \frac{1}{(A_0 - B_0)\,(A - A_0 + B_0)} = \int_0^t -r\ \mathrm{d}t', \tag{21}$$

$$\implies -\frac{1}{(A_0 - B_0)}\left[\ln(A)\right]_{A_0}^{A(t)} + \frac{1}{(A_0 - B_0)}\left[\ln(A - A_0 + B_0)\right]_{A_0}^{A(t)} = -\,rt, \tag{22}$$

$$\implies -\ln\left(\frac{A(t)}{A_0}\right) + \ln\left(\frac{A(t) - A_0 + B_0}{B_0}\right) = -\,(A_0 - B_0)rt, \tag{23}$$

$$\implies \frac{(A(t) - A_0 + B_0)\,A_0}{B_0 A(t)} = \exp(-(A_0 - B_0)rt), \tag{24}$$

$$\implies 1 - \frac{(A_0 - B_0)}{A(t)} = \frac{B_0}{A_0}\exp\left(-(A_0 - B_0)rt\right), \tag{25}$$

$$\implies A(t) = \frac{(A_0 - B_0)}{1 - \frac{B_0}{A_0}\exp\left(-(A_0 - B_0)rt\right)}. \tag{26}$$

$B(t)$ can then simply be obtained from equation (12),

$$B(t) = A(t) + B_0 - A_0, \tag{27}$$

$$= \frac{(A_0 - B_0)}{1 - \frac{B_0}{A_0}\exp\left(-(A_0 - B_0)rt\right)} + B_0 - A_0, \tag{28}$$

$$= (A_0 - B_0)\left(\frac{1}{1 - \frac{B_0}{A_0}\exp\left(-(A_0 - B_0)rt\right)} - 1\right), \tag{29}$$

$$= (A_0 - B_0)\left(\frac{B_0\exp\left(-(A_0 - B_0)rt\right)}{A_0 - B_0\exp\left(-(A_0 - B_0)rt\right)}\right). \tag{30}$$

Considering equations (26) and (30) we see that $A(0) = A_0$, $B(0) = B_0$, as required and that $A(t) \to A_0 - B_0$, $B(t) \to 0$ as $t \to \infty$. A sketch of $A(t)$ and $B(t)$ can be found in Figure 4.

13. Early warfare was simply a numbers game. The largest army would win, with losses equal to the size of the other army. Essentially, you were trading lives one for one. Captain America would not be pleased.

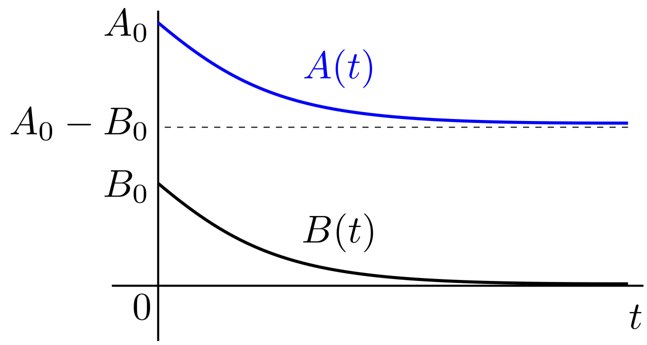

Figure 4: Solutions of equations (4) and (5).

2 Non-dimensionalisation and interpretation

Consider the following set of interaction equations for two animal populations N and P.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right) - bNP, \tag{31}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = ebNP - mP. \tag{32}$$

where b, e, K, r and m are positive constants and the initial conditions are N_0 and P_0, respectively.

1. What type of interaction is occurring between N and P?

2. Write down a set of interaction equations that could give rise to equations (31) and (32).

3. Non-dimensionalise equations (31) and (32) to get the form

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = c_1 u\,(1 - u) - uv, \tag{33}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = c_2 uv - v, \tag{34}$$

   where c_1 and c_2 should be defined explicitly in terms of K, b, e, r and m.

4. Provide an interpretation of c_1 and c_2.

5. Suppose the predator is a pest and we want to wipe the predator out, whilst maintaining a positive prey population (*e.g.* consider a fox and hen scenario). What strategy best achieves this, in terms of altering the sizes of c_1 and c_2? Explicitly, is it best to:

   - increase c_1?
   - increase c_2?
   - increase both?
   - decrease c_1?
   - decrease c_2?
   - decrease both?
   - increase c_1 and decrease c_2?
   - decrease c_1 and increase c_2?

   Explain your choice.

6. Having chosen a strategy consider how best to enact the changes of c_1 and/or c_2 in terms of altering b, e, K, r and m. What are the pros and cons? What would your advice be of best strategy?

2.1 Answers

1. N and P are undergoing a predator-prey reaction, with N as the prey and P as the predator. This is because whenever N and P interact through the NP term the N population decreases, whilst the P population increases. In the absence of predator, N will manage its population logistically, whilst in the absence of prey, P will die out.

2. Note the following are not unique, so if you have a variation on those below, as long as they still work that is fine.

$$N \underset{r/K}{\overset{r}{\rightleftarrows}} N + N, \tag{35}$$

$$P \overset{m}{\to} \emptyset, \tag{36}$$

$$N + P \overset{b}{\to} (1+e)P. \tag{37}$$

3. Let $N = [N]u, P = [P]v$ and $t = [t]\tau$. Where u, v and τ are non-dimensional and $[N]$, $[P]$ and $[t]$ are dimensional constants. Substituting these into equations (31) and (32) we derive

$$\frac{[N]}{[t]}\frac{\mathrm{d}u}{\mathrm{d}\tau} = r[N]u\left(1 - \frac{[N]u}{K}\right) - b[N][P]uv, \tag{38}$$

$$\frac{[P]}{[t]}\frac{\mathrm{d}v}{\mathrm{d}\tau} = eb[N][P]uv - m[P]v. \tag{39}$$

Rearranging we get

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = [t]ru\left(1 - \frac{u[N]}{K}\right) - b[t][P]uv, \tag{40}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = eb[N][t]uv - m[t]v. \tag{41}$$

Comparing equations (40) and (41) to equations (33) and (34) we see that we want to fix $[N] = K$, $[t] = 1/m$ and $[P] = 1/(b[t]) = m/b$. From these we can see that $c_1 = [t]r = r/m$ and $c_2 = eb[N][t] = ebK/m$.

4. Since m is the predator death rate and r is the reproduction time scale $c_1 = r/m$ is, thus, non-dimensional as it is a ratio of time-scales. Critically, it measures the system's robustness to change. For example, if c_1 is very large then r is very large (meaning a high birth rate), or m is very small (low death rate). In this case perturbations to the systems will have small effects. Alternatively is c_1 is small (*i.e.* small birth rate, or large death rate) the system is much more fragile to interference.

   eb is the gain of predator from every predator-prey interaction, m is predator death rate, thus, eb/m is a net measure of how many predators are gained in every interaction. K is carrying capacity of the domain, *i.e.* the maximum number of prey. Thus $c_2 = eb/m \times K$ is the *maximum predator fecundity*, it is a measure of how large the predator population can be.

5. Considering the interpretations of c_1 and c_2 from the previous question we see that to reduce the predator population we want to reduce c_2. Altering c_1 simply changes the time scale on which the alterations happen.

6. Since $c_2 = ebK/m$ then to reduce the predator population we can either decrease e, b, or K, or increase m. Critically, reducing K would reduce the carrying capacity of the prey, which we do not want to do. Thus, we are left with altering e, b, or m. Considering each of these in turn,

   - decreasing e represents reducing the ability of the predator to turn food into reproduction. One unlikely way of achieving this is reduce the nutritional value of the prey. Not only would this be difficult, but it will probably also reduce the profitability of our prey stock. An alternative and more widely used way is releasing sterilised predators into the wild. Although the predators will continue to pre-date, they will not be able to reproduce and, thus, we can drive the predators to extinction. However, this is often a slow and costly strategy.
   - decreasing b represents reducing predation encounters. Thus, we could increase security around the prey, essentially isolating them from attack. This is one reason why battery farming is cheaper, namely fewer chickens are lost to wild animal attacks. This is often a cheaper strategy than above, but we have to question the humanity of isolating the animal from its natural habitat.
   - increasing m represents killing the predator, perhaps through poison, or active predator hunting. Again, we have to question the humanity of this option as well as the cost-benefit when compared with decreasing b.

   As for best strategy? As long as you have considered some of the above points, and maybe others you can think of, then any defensible position will be based on the specifics of a given case. For example, it is easier to wipe out a pest like termites than it is to wipe out a fox population. Equally, mixed strategies are often the most beneficial, namely, sterilisation alongside culling, alongside prey protection.

3 Using ode45

As in MA0232 I will be offering you the chance to have a go at understanding Matlab. It is free to download, all you need is follow this (clickable) link `https://uk.mathworks.com/academia/tah-support-program/eligibility.html` and sign up using your Cardiff email address.

Throughout this course we will be simulating ODEs of many types and, thus, it is good to be able to check your answers. The more you simulate, the better your intuition. In order to simulate ODEs we are going to use Matlab's inbuilt command `ode45`. Matlab has several ODE solvers, but `ode45` is the most general "catch all" code. Essentially, you try it first and if it does not work you try something more specific. However, for our purposes it will be adequate.

For a high level description of `ode45` type `help ode45`, or `doc ode45` into Matlab and it will bring up a brief description and a full description, respectively. Here, I will provide a brief overview of a general code that will work on most systems throughout this course.

The code below should be available as `General_ode45_code.m` and it solves problem 1. Either download it directly, or copy and paste it into Matlab. The green text are comments to help you understand what is going on. Essentially, for different ODE systems you should only really have to change the description in lines 49-50. Note you may also need to change the length of time over which the equations are solved (`Tend`).

```
%% Ensure we start from a blank slate
clear all
close all
clc

%% Initialise variables
fs=15; % Set fontsize

Tend=0.3; % Set final time
N=100; % Set number of steps
t=linspace(0,Tend,N); % Set number of time points at which the solution is to be evaluated
IC=[200;
    190]; %Initial conditions as a column vector

%% Solve ODE

% Each component is a row in the matrix y.
[t,y] = ode45(@ODE_function,t,IC);

%% Plot Solution

%%% (time,solution plots)
hold on % Allows us to plot multiple graphs at once
plot(t,y(:,1)) % Plots the first solution
plot(t,y(:,2)) % Plots the second solution
legend('u','v') % Adds a legend and defines the lines for clarity

% Label axes for clarity
xlabel('t')
ylabel('Solutions')

set(gca,'fontsize',fs) % Changes the axis fontsize

%%% Phasespace plots
figure % Starts a new figure without overwriting the previous one
plot(y(:,1),y(:,2))
xlabel('u')
ylabel('v')
set(gca,'fontsize',fs)




%% ODE description
function dydt=ODE_function(t,y)
r=1; %Interaction parameter

% Defining the ODEs as a column vector. Each row is one of the ODEs.
dydt=[-r*y(1)*y(2) %Line 49
      -r*y(1)*y(2)]; %Line 50

end
```

Once you have had a play with the above code try and adapt it to solve equations (33) and (34).

Exam Revision

Problem sheets 3-5 of course MA0232 contain many examples and solutions that will provide adequate practice at: deriving steady states, non-dimensionalisation, characterising stability and sketching graphs.